On the Strong Roman Domination Number of Graphs

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Abstract

Based on the history that the Emperor Constantine decreed that any undefended place (with no legions) of the Roman Empire must be protected by a "stronger" neighbor place (having two legions), a graph theoretical model called Roman domination in graphs was described. A Roman dominating function for a graph $G = (V, E)$, is a function $f : V \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ has at least a neighbor w in G for which $f(w) = 2$. The Roman domination number of a graph is the minimum weight, $\sum_{v\in V} f(v)$, of a Roman dominating function. In this paper we initiate the study of a new parameter related to Roman domination, which we call strong Roman domination number and denote it by $\gamma_{StR}(G)$. We approach the problem of a Roman domination-type defensive strategy under multiple simultaneous attacks and begin with the study of several mathematical properties of this invariant. In particular, we first show that the decision problem regarding the computation of the strong Roman domination number is NP-complete, even when restricted to bipartite graphs. We obtain several bounds on such a parameter and give some realizability results for it. Moreover, we prove that for any tree T of order $n \geq 3$, $\gamma_{StR}(T) \leq 6n/7$ and characterize all extremal trees.

Keywords: Domination; Roman domination; Roman domination number; strong Roman domination.
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1 Introduction

The concept of Roman domination in graphs was introduced by Cockayne *et al.* [9], according to some connections with historical problems of defending the Roman Empire described in [21, 24]. A *Roman dominating function* (RDF for short) on a graph $G = (V, E)$ is defined as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. A $\gamma_R(G)$-*function* is a Roman dominating function of G with weight $\gamma_R(G)$. After this seminal work [9], several investigations have been focused into obtaining properties of this invariant [12, 13, 16, 17, 26].

On the other hand, in order to generalize or improve some particular property of the Roman domination in its standard presentation, some variants of Roman domination have been introduced and frequently studied. Those variants are frequently related to modifying the conditions in which the vertices are dominated, or to adding an extra property to the Roman domination property itself. For instance we remark here variants like the following ones: independent Roman domination [1, 8], edge Roman domination [22], weak Roman domination [10, 18], total Roman domination[1] [20], signed Roman domination [4, 23], signed Roman edge domination [3], Roman k-domination [15, 19] and distance Roman domination [5], among others. On the other hand, an interesting version regarding the defense of the "Roman Empire" against multiple attacks was described in [17]. In this article we propose a new version of Roman domination in which we also deal with multiple attacks.

To begin with our work, we first introduce the terminology and notation we shall use throughout the exposition. Unless stated on the contrary, other notation and terminology not explicitly given here could be find in [7]. Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$ and the *size* $|E|$ of G is denoted by $m = m(G)$. By $u \sim v$ we mean that u, v are adjacent, *i.e.*, $uv \in E$. For a non-empty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors v has in X, or equivalently, $N_X(v) = \{u \in X : u \sim v\}$. In the case $X = V$, we use only $N(v)$, instead of $N_V(v)$, which is also called the *open neighborhood* of the vertex $v \in V$. The *close neighborhood* of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$. For any vertex v, the cardinality of $N(v)$ is the *degree* of v in G, denoted by $\deg_G(v)$ (or just $\deg(v)$ if there is no confusion). The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *universal vertex* of G is a vertex which is adjacent to every other vertex of G.

A *uv-path* in G, joining the (end) vertices $u, v \in V$, is a finite alternating sequence: $u_0 = u, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$ of different vertices and edges, beginning with the vertex u and ending with the vertex v, so that $e_i = u_{i-1}u_i$ for all $i = 1, 2, \ldots, k$. The number of edges in a path is called the *length* of the path. The length of a shortest uv-path is the *distance between the vertices* u and v, and it is denoted by $d(u, v)$. The maximum among all the distances between two vertices in a graph G is denoted by $Diam(G)$, the *diameter* of G. A *cycle* is a uu-path. The *girth* of a graph G, denoted by $g(G)$, is the length of its shortest cycle. The girth of a graph with no

[1] The concept of total Roman domination was introduced in [20] albeit in a more general setting. Its specific definition has appeared in [2].

cycle is defined ∞.

The set of vertices $D \subset V$ is a *dominating set* if every vertex v not in D is adjacent to at least one vertex in D. The minimum cardinality of any dominating set of G is the *domination number* of G and is denoted by $\gamma(G)$. A dominating set D in G with $|D| = \gamma(G)$ is called a $\gamma(G)$-set. Notice that a graph having a universal vertex has domination number equal to one.

Let f be a Roman dominating function for G and let $V(G) = B_0 \cup B_1 \cup B_2$ be the sets of vertices of G induced by f, where $B_i = \{v \in V \ : \ f(v) = i\}$, for all $i \in \{0, 1, 2\}$. It is clear that for any Roman dominating function f of a graph G, we have that $f(V) = \sum_{u \in V} f(u) = 2|B_2| + |B_1|$. A Roman dominating function f can be represented by the ordered partition (B_0, B_1, B_2) of $V(G)$. It is proved that for any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ [9]. Note that if $C_1, C_2, \ldots, C_t$ are the components of G, then $\gamma_R(G) = \sum_{i=1}^{t} \gamma_R(C_i)$. Therefore, from now on we will only consider connected graphs, unless it would be necessary to satisfy some specific condition.

The defensive strategy of Roman domination is based in the fact that every place in which there is established a Roman legion (a label 1 in the Roman dominating function) is able to protect itself under external attacks; and that every unsecured place (a label 0) must have at least a stronger neighbor (a label 2). In that way, if an unsecured place (a label 0) is attacked, then a stronger neighbor could send one of its two legions in order to defend the weak neighbor vertex (label 0) from the attack. Two examples of Roman dominating functions are depicted in Figure 1.

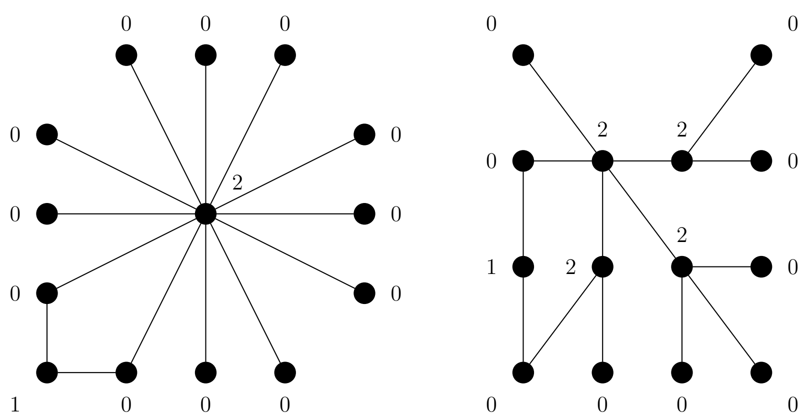


Figure 1: Two Roman dominating functions.

Although these two functions (Figure 1) satisfy the conditions to be Roman dominating functions, they correspond to two very different real situations. The unique strong place (2) in the left hand side graph must defend up to 12 weak places from possible external attacks. However, in the right hand side graph, the task of defending the unsecured places is divided into several strong places. This observation has led us to pose the following question: how many weak places may defend a strong place having two legions? Taking into account that the strong place must leave one of its legions to defend itself, the situation depicted on the left hand side graph of Figure 1 seems to be a not efficient defensive strategy: the Roman domination strategy fails against a "multiple attack" situation. If several simultaneous attacks to weak places are developed, then the only stronger place will be not able to defend its neighbors efficiently. With this motivation in mind, we introduce the concept of strong Roman dominating function as follows. For our purposes, we consider that a strong place should be able to defend itself and, at least half of its weak neighbors.

Consider a graph G of order n and maximum degree Δ. Let $f : V(G) \rightarrow \{0, 1, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ be a function that labels the vertices of G. Let $B_j = \{v \in V : f(v) = j\}$ for $j = 0, 1$ and let $B_2 = V \setminus (B_0 \cup B_1) = \{v \in V : f(v) \geq 2\}$. Then, f is a *strong Roman dominating function* (StRDF for short) for G, if every $v \in B_0$ has a neighbor w, such that $w \in B_2$ and $f(w) \geq 1 + \lceil \frac{1}{2}|N(w) \cap B_0| \rceil$. In Figure 2 we show a strong Roman dominating function for each of the graphs shown in Figure 1. The minimum weight, $w(f) = f(V) = \sum_{u \in V} f(u)$, over all the strong Roman dominating functions for G, is called the *strong Roman domination number* of G and we denote it by $\gamma_{StR}(G)$. An StRDF of minimum weight is called a $\gamma_{StR}(G)$-function.

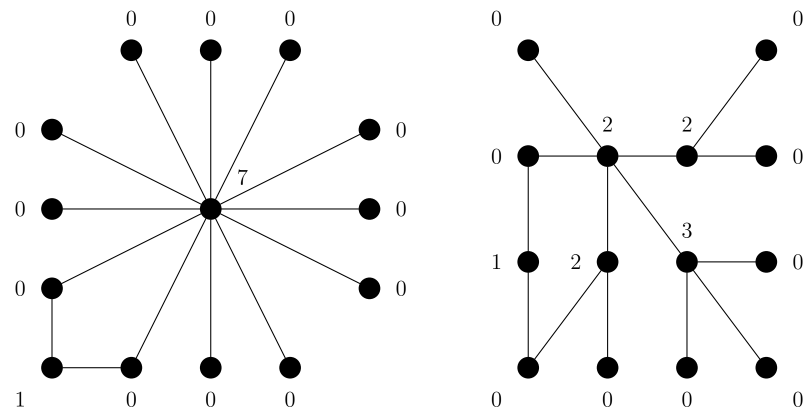


Figure 2: Two strong Roman dominating functions.

The relationship between the vertices having label two (2) in a Roman dominating function, and those ones having labels with value greater than one in a strong Roman dominating function is not exactly clear, as we can observe throughout the following examples. For instance, a minimum Roman dominating function is shown on the left hand side of Figure 3. However, if we modify the labels with value two (2) to labels with value four (4), then a strong Roman dominating function is obtained, but it has not minimum weight. The left hand side of Figure 4 shows a minimum strong Roman dominating function. Nevertheless, if the vertices with label three (3) are changed to a label with value two (2), then a Roman dominating function is obtained, but it has not minimum weight.

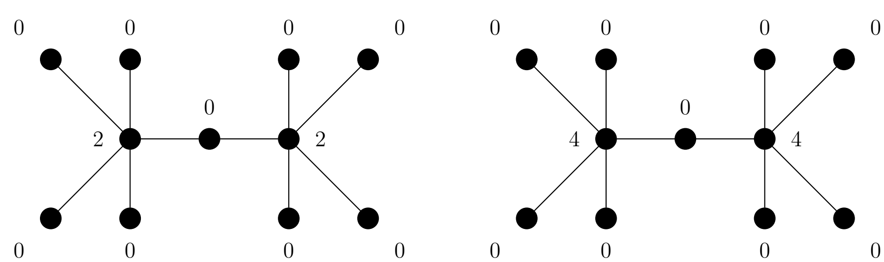


Figure 3: A minimum Roman dominating function does not "generate" a minimum strong Roman dominating function.

We make use of the following results in this section, some of which are already published or straightforward to observe and so, we omit their proofs.

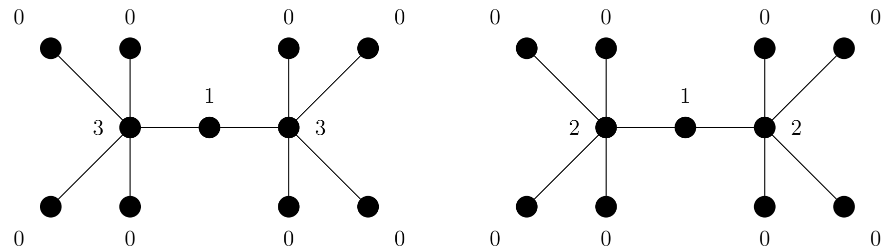


Figure 4: A minimum strong Roman dominating function does not "generate" a minimum Roman dominating function.

Theorem A. [6] *For any tree T of order $n \geq 3$, $\gamma_R(T) \leq 4n/5$.*

Theorem B. [6] *For paths P_n and cycles C_n, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$.*

Observation 1. *For any connected graph G with $\Delta(G) \leq 2$, $\gamma_{StR}(G) = \gamma_R(G)$.*

Based on the remark above, from now on, in this work we focus mainly on graphs with maximum degree $\Delta \geq 3$.

2 The complexity of the strong Roman domination problem

In this section we deal with the following decision problem. We must remark that results obtained here are a generalization of that results previously obtained in [12].

STRONG ROMAN DOMINATION PROBLEM INSTANCE: A non-trivial graph G and a positive integer r PROBLEM: Deciding whether the strong Roman domination number of G is less than r

For our purposes of studying the complexity of the STRONG ROMAN DOMINATION PROBLEM (StRD-Problem for short) we will use the following variation of the 3-SAT problem which was proved to be NP-complete in [11]. Let $\mathcal{F}$ be a boolean formula with set of variables U and set of clauses C. The clause-variable graph of $\mathcal{F}$ is defined as the graph $G_{\mathcal{F}}$ with vertex set $V = U \cup C$ and edge set $E = \{(v, c) : v \in V, c \in C \text{ and } v \in c\}$.

Lemma 2. [11] *The problem of deciding whether a boolean formula $\mathcal{F}$ is satisfiable is NP-complete, even if*

- *every variable occurs exactly once negatively and once or twice positively,*
- *every clause contains two or three distinct variables,*
- *every clause with three distinct variables contains at least one negative literal, and*
- *$G_{\mathcal{F}}$ is planar.*

In [11], the problem above was called 1-Negative Planar 3-SAT. If there are two equal clauses in a boolean formula $\mathcal{F}$, then we can reduce such a formula to other one having all its clauses unique and this does not change the veracity of the formula. Also, we can consider that the number of clauses in the boolean formula is greater than or equal to the number of variables. Therefore, from now on, we will consider a boolean formula on n variables and m pairwise different clauses, with $m \geq n$, and satisfying the conditions of Lemma 2. Such a boolean formula will be represented by $\mathcal{F}_3$.

To prove that SRD-Problem is NP-complete, we present a reduction from 1-Negative Planar 3-SAT. The outline of the procedure behind this reduction is the following. We begin with an instance of 1-Negative Planar 3-SAT, that is a boolean formula $\mathcal{F}_3$. We consider a planar embedding of its clause-variable graph $G_{\mathcal{F}_3}$, and replace each variable vertex of $G_{\mathcal{F}_3}$ by a variable gadget, and each clause vertex of $G_{\mathcal{F}_3}$ by a clause gadget. Hence, we identify the vertices of the variable gadgets and the vertices of the clause gadgets in its "corresponding" way and, in this sense, we obtain a planar graph $G_{\mathcal{F}_3}$ which we will use as our instance of SRD-Problem.

We consider the following variable gadgets and clause gadgets. Let $X = \{a_1, a_2, ..., a_n\}$ (the variables) and $C = \{C_1, C_2, \dots, C_m\}$ (the clauses) be an arbitrary instance of 1-Negative Planar 3-SAT with $m \geq n$. The literals are denoted by a_i (for positive) or $\overline{a_i}$ (for negative). Every clause C_i, is represented by a vertex denoted by c_i. Every variable a_i is represented by a complete bipartite graph $H_i \cong K_{2,3}$, with partite sets $A_i = \{a_i, \overline{a_i}\}$ (each one for the corresponding literals of a_i) and $B_i = \{x_i, y_i, z_i\}$. To construct our graph G_{F_3}, we add the edge $a_i\overline{a_i}$, and the two or three edges connecting each clause vertex c_i with the vertices corresponding to the literals in the clause C_i. In order to be used while proving our results, since $m \geq n$, we consider a partition of the vertex set of G_{F_3} into n sets $S_i = V(H_i) \cup \{c_j\}$, where either $a_i \in C_j$ or $\overline{a_i} \in C_j$ and a set $Y = V(G_{F_3}) - (\bigcup_{i=1}^{n} S_i)$, given by those vertices c_l not belonging to any B_i (notice that this Y could be empty, in the case $m = n$). Also notice that $|S_i| = 6$ for every $i \in \{1, ..., n\}$. We must point out that the same construction was already used by Paul A. Dreyer [12] in his Ph. D. thesis, to study the complexity of the standard Roman domination, although the procedure of using it was slightly different.

We will prove that a boolean formula $\mathcal{F}_3$ on n variables and m clauses, being an instance of 1-Negative Planar 3-SAT, has a satisfying truth assignment if and only if the graph $G_{\mathcal{F}_3}$ satisfies $\gamma_{StR}(G_{\mathcal{F}_3}) = 4n$. Notice that $G_{\mathcal{F}_3}$ has order $m + 5n$ and size at most $3m + 7n$. Moreover, every vertex c_i has degree two or three (since every clause has two or three literals), every vertex a_i has degree five or six, and every $\overline{a_i}$ has degree five (since each variable appears in $\mathcal{F}_3$ exactly once negatively and once or twice positively). Figure 5 shows an example for the case $\mathcal{F}_3 = (a_1 \vee a_2 \vee \overline{a_3}) \wedge (\overline{a_1} \vee a_3) \wedge (a_1 \vee \overline{a_2} \vee a_3)$. Next we observe some other properties of the graph $G_{\mathcal{F}_3}$.

Remark 3. *$G_{\mathcal{F}_3}$ is planar.*

Proof. Since all the clauses of $\mathcal{F}_3$ are distinct, not any copy of the complete bipartite graph $K_{3,3}$ is a subgraph of $G_{\mathcal{F}_3}$. On the other hand, not any subgraph of $G_{\mathcal{F}_3}$ is isomorphic to the complete graph K_5. Therefore, by the Kuratowski's theorem is obtained that $G_{\mathcal{F}_3}$ is planar. □

Remark 4. *For every subgraph of $G_{\mathcal{F}_3}$ induced by S_i, with $i \in \{1, ..., n\}$ and every $\gamma_{StR}(G_{\mathcal{F}_3})$-function f, it follows $f(S_i) \geq 4$.*

Proof. Notice that the vertices of the set $B_i \subset S_i$ are pairwise independent and they have the same shared neighbors (the two vertices of A_i). Let us suppose that $f(S_j) \leq 3$ for some $j \in \{1, ..., n\}$.

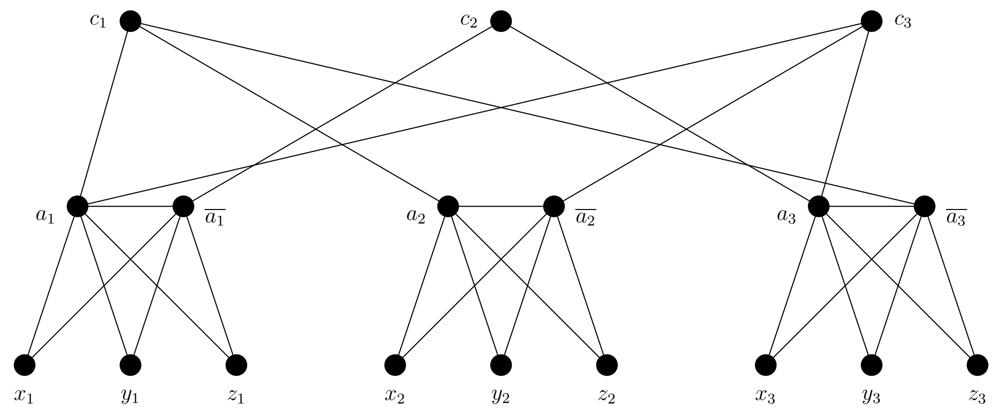


Figure 5: The graph $G_{\mathcal{F}_3}$ where $\mathcal{F}_3 = (a_1 \vee a_2 \vee \overline{a_3}) \wedge (\overline{a_1} \vee a_3) \wedge (a_1 \vee \overline{a_2} \vee a_3)$.

Since $|S_j| \geq 6$, at least three vertices of S_j have label zero (0) by f and two of them cannot be both in A_j. As a consequence, at least one vertex of B_j has label zero (0) and at least one vertex $u \in \{a_j, \overline{a_j}\} = A_j$, has label at least two (2) by f. In this sense, let $t \in \{3, 4, 5\}$ be the number of vertices in S_j with label zero (0) by f. So, we have that $f(u) \geq \lceil \frac{t}{2} \rceil + 1$. Therefore, it follows that

$$f(S_j) \geq f(u) + f(A_j - \{u\}) + f(B_j) \geq \left\lceil \frac{t}{2} \right\rceil + 1 + 5 - t = 6 - \left\lfloor \frac{t}{2} \right\rfloor \geq 4,$$

a contradiction. Therefore, $f(S_i) \geq 4$ for every $i \in \{1, ..., n\}$. $\square$

Since, any boolean formula $\mathcal{F}_3$ has n variables, the result above leads to the following corollary.

Corollary 5. *For any boolean formula $\mathcal{F}_3$ on n variables and m clauses with $m \geq n$, being an instance of 1-Negative Planar 3-SAT, $\gamma_{StR}(G_{\mathcal{F}_3}) \geq 4n$.*

Now we present the main result of this section.

Theorem 6. *Let $\mathcal{F}_3$ be a formula on n variables and m clauses, being an instance of 1-Negative Planar 3-SAT. Then $\gamma_{StR}(G_{\mathcal{F}_3}) = 4n$ if and only if $\mathcal{F}_3$ is satisfiable.*

Proof. We assume that $\mathcal{F}_3$ has satisfying truth assignment. That is, for any variable a_i, we have either a_i or $\overline{a_i}$ has assigned the value True. We will define a function g on $V(G_{\mathcal{F}_3})$ in the following way. If a_i has the value True, then we define $g(a_i) = 4$. On the contrary, if a_i has the value False, then we define $g(\overline{a_i}) = 4$. For any other vertex $w \in V(G_{\mathcal{F}_3})$, we define $g(w) = 0$. Since the definition of this function is based on the satisfying truth assignment of $\mathcal{F}_3$, it is straightforward to observe that the function g is a strong Roman dominating function of weight $w(g) = 4n$. Thus, $\gamma_{StR}(G_{\mathcal{F}_3}) \leq 4n$ and, by Corollary 5, we have that $\gamma_{StR}(G_{\mathcal{F}_3}) = 4n$.

On the other hand, we assume that $\gamma_{StR}(G_{\mathcal{F}_3}) = 4n$. Since every vertex a_i has degree five or six, and every $\overline{a_i}$ has degree five (each variable appears in $\mathcal{F}_3$ exactly once negatively and once or twice positively), there exists a $\gamma_{StR}(G_{\mathcal{F}_3})$-function h such that for every $i \in \{1, ..., n\}$, either $h(a_i) = 4$ or $h(\overline{a_i}) = 4$, and any other vertex of $G_{\mathcal{F}_3}$ has label zero (0) by h. Now, if $h(a_i) = 4$, then we set a_i as True. On the contrary, if $h(\overline{a_i}) = 4$, then we set $\overline{a_i}$ as False. Since every vertex c_j corresponding to a clause (notice that it has label zero) is adjacent to at least one vertex with label four (4), it follows that the clause is satisfied. Therefore, the formula $\mathcal{F}_3$ has a satisfying truth assignment. $\square$

As a consequence of Theorem 6 and Remark 3 we have the following result, which completes the proof of the NP-completeness of the SRD-Problem.

Corollary 7. *STRONG ROMAN DOMINATION PROBLEM is NP-complete, even when restricted to planar graphs.*

3 Preliminary bounds on the strong Roman domination number

According to the NP-completeness of the SRD-Problem, it is therefore desirable to find sharp bounds for the strong Roman domination number of graphs. In this section, we establish some sharp bounds for the strong Roman domination number of graphs in terms of several parameters of the graph.

Proposition 8. *Let G be a graph of order n. Then*

$$\gamma_R(G) \le \gamma_{StR}(G) \le \left(1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil\right) \gamma(G).$$

Proof. Let f be a $\gamma_{StR}(G)$-function. Define $\widetilde{f} : V(G) \to \{0, 1, \ldots, \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\}$ by $\widetilde{f}(v) = 2$ for $v \in B_2$ and $\widetilde{f}(v) = f(v)$ otherwise. It is straightforward to observe that $\widetilde{f}$ is a Roman dominating function of G and hence $\gamma_R(G) \le w(\widetilde{f}) \le w(f) = \gamma_{StR}(G)$

To prove the upper bound, let D be a dominating set of minimum cardinality and define $h : V(G) \to \{0, 1, \ldots, \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\}$ by $h(v) = 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil$ for $v \in D$ and $h(v) = 0$ otherwise. Obviously, h is a strong Roman dominating function for G and, as a consequence, $\gamma_{StR}(G) \le w(h) = \left(1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil\right) \gamma(G)$. □

Proposition 9. *Let G be a graph of order n. Then*

$$\gamma_{StR}(G) \le n - \left\lfloor \frac{\Delta(G)}{2} \right\rfloor.$$

Proof. Let $v \in V(G)$ be a vertex of maximum degree $\Delta(G)$ and define $f : V(G) \to \{0, 1, \ldots, \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\}$ by $f(v) = 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil$, $f(x) = 0$ for $x \in N(v)$ and $f(x) = 1$ otherwise. Clearly, f is a strong Roman dominating function for G and so $\gamma_{StR}(G) \le w(f) = 1 + \left\lceil \frac{\Delta}{2} \right\rceil + n - \Delta - 1 = n - \left\lfloor \frac{\Delta}{2} \right\rfloor$. This completes the proof. □

An immediate consequence of Observation 1 and Proposition 9 now follows.

Corollary 10. *Let G be a connected graph or order n. Then $\gamma_{StR}(G) = n$ if and only if $G =$* K_1 or K_2.

The two propositions above give two different upper bounds on $\gamma_{StR}(G)$ which are involving the maximum degree Δ of the graph. So, an interesting question regarding this could be: Can we compare them with respect to its efficiency? As we can observe in the next two examples, such a question could be not completely clearly addressed.

Notice that the left hand side graph of Figure 6 satisfies that $\left(1+\left\lceil\frac{\Delta}{2}\right\rceil\right)\gamma(G)=6<7=n-\left\lfloor\frac{\Delta}{2}\right\rfloor$. Nevertheless, the right hand side graph shown in Figure 6 carries out that $n-\left\lfloor\frac{\Delta}{2}\right\rfloor=4<6=\left(1+\left\lceil\frac{\Delta}{2}\right\rceil\right)\gamma(G)$.

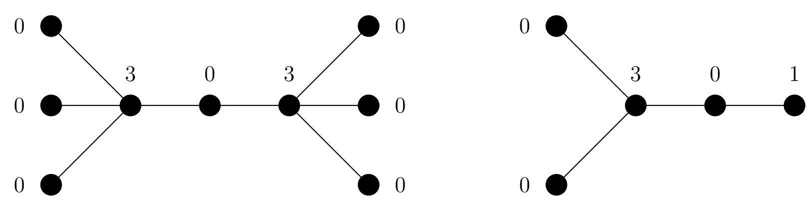


Figure 6: Graph used to compare upper bound of Propositions 8 and 9.

A *rooted graph* is a graph in which one vertex is labeled in a special way so as to distinguish it from other vertices. The special vertex is called the *root* of the graph. Let G be a labeled graph on n vertices. Let $\mathcal{H}$ be an ordered sequence of n rooted graphs H_1, H_2,...,H_n. The *rooted product graph* $G(\mathcal{H})$ is the graph obtained by identifying the root of H_i with the i^{th} vertex of G [14]. If the family $\mathcal{H}$ consists of n isomorphic rooted graphs, then we use the notation $G\circ_v H$. Moreover, if H is a vertex transitive graph, then $G\circ_v H$ does not depend on the choice of v, up to isomorphism. In such a case we will just write $G\circ H$.

Proposition 11. *For any connected graph G on n vertices,*

$$\gamma_{StR}(G)\leq n-\left\lfloor\frac{1+\operatorname{diam}(G)}{3}\right\rfloor.$$

Furthermore, this bound is sharp for the rooted product graph $P_3\circ P_2$.

Proof. Let $P=v_1v_2\dots v_q$, $q=\operatorname{diam}(G)+1$, be a diametral path in G and let f be a $\gamma_{StR}(P)$-function. Define $g:V(G)\to\{0,1,\dots,\left\lceil\frac{\Delta(G)}{2}\right\rceil+1\}$ by $g(x)=f(x)$ for $x\in V(P)$ and $g(x)=1$ otherwise. Obviously g is a strong Roman dominating function of G. Hence,

$$\gamma_{StR}(G)\leq\omega(f)+(n-\operatorname{diam}(G)-1)=n-\left\lfloor\frac{1+\operatorname{diam}(G)}{3}\right\rfloor.$$

$\square$

Proposition 12. *Let G be a connected graph of order n with $g(G)\geq 3$. Then*

$$\gamma_{StR}(G)\leq n-\left\lfloor\frac{g(G)}{3}\right\rfloor.$$

Furthermore, the bound is sharp for the rooted product graph $C_3\circ P_2$.

Proof. Assume C is a cycle of G with $g(G)$ edges. Let f be a $\gamma_{StR}(C)$-function and define $g : V(G) \to \{0, 1, \dots, \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1\}$ by $g(x) = f(x)$ for $x \in V(C)$ and $g(x) = 1$ otherwise. Clearly g is a strong Roman dominating function of G that implies $\gamma_{StR}(G) \leq n - |V(C)| + \left\lceil \frac{2g(G)}{3} \right\rceil = n - \left\lfloor \frac{g(G)}{3} \right\rfloor$. □

Next we continue with another upper bound on the strong Roman domination number of graphs. The proof of such a bound uses a probabilistic approach, which in some sense, is a generalization of a similar result presented in [9].

Proposition 13. *Let G be a graph of order n, minimum degree δ and maximum degree Δ, such that $\lceil \frac{\Delta}{2} \rceil < \delta$. Then*

$$\gamma_{StR}(G) \leq \frac{\left(1 + \lceil \frac{\Delta}{2} \rceil\right) n}{\delta + 1} \left(\ln \left(\frac{1 + \delta}{1 + \lceil \frac{\Delta}{2} \rceil} \right) + 1 \right).$$

Proof. Given a set $A \subset V(G)$, we consider the set $B = V(G) - N[A]$. We denote by $A^c = \{v \in V(G) : v \notin A\}$. Notice that $B = A^c \cap N(A)^c = N[A]^c$. Given $v \in V(G)$, we define p as the probability that v would belong to A, $p = P[v \in A]$. We consider now $P[v \in B]$ as the probability of the event such that v does not belong to A and also, the neighbors of v are not in A. That is, $P[v \in B] = P[v \in A^c \cap N(A)^c] = (1-p)(1-p)^{\delta(v)} = (1-p)^{1+\delta(v)} \leq (1-p)^{1+\delta}$. According to this, we can approximate the expected values of $|A|$ and $|B|$: $E[|A|] = np$ and $E[|B|] = n(1-p)^{1+\delta(v)} \leq n(1-p)^{1+\delta(G)} \leq ne^{-p(1+\delta)}$. Now, define the function $f : V(G) \to \{0, 1, \dots, 1 + \lceil \frac{\Delta}{2} \rceil\}$ by

$$f(v) = \begin{cases} 1 + \lceil \frac{\Delta}{2} \rceil & \text{if } v \in A \\ 0 & \text{if } v \in N(A) \\ 1 & \text{if } v \in B. \end{cases}$$

Then, the expected value of $f(V)$ is:

$$\begin{aligned} E[f(V)] &= \left(1 + \left\lceil \frac{\Delta}{2} \right\rceil\right) E[|A|] + E[|B|] \\ &\leq \left(1 + \left\lceil \frac{\Delta}{2} \right\rceil\right) np + ne^{-p(1+\delta)}. \end{aligned}$$

The last expression attains its minimum value $\left(1 + \lceil \frac{\Delta}{2} \rceil\right) n - n(1+\delta)e^{-p(1+\delta)}$ if and only if $e^{-p(1+\delta)} = \frac{1+\lceil \frac{\Delta}{2} \rceil}{\delta+1}$. Moreover, the last equality has solution p such that $0 < p < 1$, if $\frac{1+\lceil \frac{\Delta}{2} \rceil}{\delta+1} < 1$, which means $\frac{1+\delta}{1+\lceil \frac{\Delta}{2} \rceil} > 1$. This leads to $\lceil \frac{\Delta}{2} \rceil < \delta$ and, as a consequence, the solution is $p = \frac{1}{1+\delta} \ln \left(\frac{1+\delta}{1+\lceil \frac{\Delta}{2} \rceil} \right)$. Therefore, $\gamma_{StR}(G) \leq \left(1 + \lceil \frac{\Delta}{2} \rceil\right) \frac{n}{1+\delta} \ln \left(\frac{1+\delta}{1+\lceil \frac{\Delta}{2} \rceil} \right) + \left(1 + \lceil \frac{\Delta}{2} \rceil\right) \frac{n}{1+\delta}$, which leads to the result. □

The last proposition provides a new upper bound for $\gamma_{StR}(G)$, which we can compare with some of the previous upper bounds. For instance, we could compare it with the one of Proposition 9. That is, we want to find those graphs G with

$$\frac{\left(1 + \lceil \frac{\Delta}{2} \rceil\right) n}{\delta + 1} \left(\ln \left(\frac{1 + \delta}{1 + \lceil \frac{\Delta}{2} \rceil} \right) + 1 \right) \leq n - \left\lfloor \frac{\Delta}{2} \right\rfloor. \tag{1}$$

As an example, we consider those graphs G with above property such that $\lceil \frac{\Delta}{2} \rceil = \delta - 1$. Some algebraic work on (1) leads to a graph G satisfying $\delta \leq \sqrt{n+1}$. For such a graph, the bound of Proposition 13 is better than the one in Proposition 9. Otherwise, vise versa. Similar conclusions could be extracted using other different statements, which makes that the process of comparing all these bounds above is not clear.

We observe that a strong Roman dominating defensive strategy needs, in general, more legions than a Roman dominating one, so the advantage is not to safe resources but to design an stronger empire against external attacks. Under the strong Roman dominating strategy, any strong vertex must be able to defend itself and at least one half of its weak neighbors. The goal is then to deal with this situation by using as few resources (legions) as possible.

Next result gives the minimum number of legions which are needed to protect the Roman empire fortifications under the strong Roman domination strategy.

Proposition 14. *Let G a graph of order n. Then*

$$\gamma_{StR}(G) \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

Moreover, if n is odd, then equality holds if and only if $\Delta(G) = n-1$.

Proof. Let f be a $\gamma_{StR}(G)$-function and let $B_1 = \{w \in V(G) \mid f(w) = 1\}$, $B_2 = \{w \in V(G) \mid f(w) \geq 2\}$, $B_0 = \{w \in V(G) \mid f(w) = 0\}$, $B_0^1 = \{w \in B_0 \mid |N(w) \cap B_2| = 1\}$ and $B_0^2 = B_0 \setminus B_0^1$. Clearly (B_0, B_1, B_2) is a partition of $V(G)$ and (B_0^1, B_0^2) is a partition of B_0. Hence, $n = |B_1| + |B_2| + |B_0^1| + |B_0^2| = |B_1| + |B_2| + |B_0|$. It follows that

$$\begin{aligned}
\gamma_{StR}(G) &= \sum_{v \in B_1} f(v) + \sum_{v \in B_2} f(v) \\
&\geq |B_1| + |B_2| + \frac{1}{2}|B_0^1| + \sum_{w \in B_0^2} \frac{1}{2}|N(w) \cap B_2| \\
&\geq |B_1| + |B_2| + \frac{1}{2}|B_0^1| + |B_0^2| \\
&= n - \frac{1}{2}|B_0^1| \\
&\geq n - \frac{n-1}{2} \qquad (\text{since } |B_0^1| \leq n-1) \\
&= \frac{n+1}{2}.
\end{aligned}$$

Therefore, the result follows, since $\gamma_{StR}(G)$ is an integer number.

If n is odd and $\Delta(G) = n-1$, then we deduce from Proposition 9 that $\gamma_{StR}(G) \leq n - \left\lfloor \frac{\Delta(G)}{2} \right\rfloor = \frac{n+1}{2}$ and so $\gamma_{StR}(G) = \frac{n+1}{2}$.

Conversely, let n is odd and $\gamma_{StR}(G) = \frac{n+1}{2}$. Then the inequalities occurring in the proof become equalities. Hence $|B_0^1| = n-1$ and $|B_2| = 1$ that implies $\Delta(G) = n-1$ and the proof is completed. $\square$

Next result is an immediate consequence of Propositions 9 and 14.

Corollary 15. *For $n \geq 1$, $\gamma_{StR}(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$.*

4 Trees

In this section we first show that for any tree T of order $n \geq 3$, $\gamma_{StR}(T) \leq \frac{6n}{7}$ and then, we characterize all extremal trees which attain this upper bound. To begin with, we need to introduce some terminology and notation. A vertex of degree one is called a *leaf*, and its neighbor is called a *stem*. If v is an stem, then L_v will denote the set of all leaves adjacent to v. An stem v is called *end-stem* if $|L_v| > 1$. For $r, s \geq 1$, a *double star* $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. For a vertex v in a rooted tree T, let $C(v)$ denotes the set of children of v, $D(v)$ denotes the set of descendants of v and $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, $\text{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v.

A *subdivision* of an edge uv is obtained by replacing the edge uv with a path uwv, where w is a new vertex. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider* $S_{t,t}$. A *wounded spider* $S_{t,q}$ is the graph formed by subdividing q of the edges of a star $K_{1,t}$ for $t \geq 2$ where $q \leq t-1$. Note that stars are wounded spiders. A *spider* is a healthy or wounded spider. We now present a result on the strong Roman domination of double stars. Notice that spiders and double stars can be also represented as rooted product graphs.

Lemma 16. *For any integers $1 \leq p \leq q$ and any double star $T = S(p, q)$ of order $n = p + q + 2$, $\gamma_{StR}(T) < \frac{6n}{7}$.*

Proof. Let u, v be the non-central vertices of T. If $q = 1$, then also $p = 1$. So, $T = P_4$ and we have $\gamma_{StR}(P_4) = 3 < \frac{6n}{7}$. Assume that $q \geq 2$. First let $p = 1$. Define f on $V(T)$ by assigning $1 + \lceil \frac{q+1}{2} \rceil$ to v, 1 to the leaf at distance 2 from v and 0 to the other vertices. Obviously f is a StRDF of T of weight $\lceil \frac{n+2}{2} \rceil$ and we have $\gamma_{StR}(T) = \lceil \frac{n+2}{2} \rceil < \frac{6n}{7}$ because $n \geq 5$. Now let $p \geq 2$. Define f on $V(T)$ by assigning $1 + \lceil \frac{q}{2} \rceil$ to v, $1 + \lceil \frac{p}{2} \rceil$ to u, and 0 to the remaining vertices. Obviously f is a StRDF of T of weight $2 + \lceil \frac{p}{2} \rceil + \lceil \frac{q}{2} \rceil$. Considering the parity of p and q, it is straightforward to see that $\gamma_{StR}(T) = 2 + \lceil \frac{p}{2} \rceil + \lceil \frac{q}{2} \rceil < \frac{6n}{7}$ and the proof is complete. $\square$

The following bound is the main result of this section, where we bound the strong Roman domination of trees in terms of its order.

Theorem 17. *If T is a tree of order $n \geq 3$, then*

$$\gamma_{StR}(T) \leq \frac{6n}{7}.$$

Proof. We proceed by induction on $n \geq 3$. The statement holds for all trees of order $n \leq 5$. For the inductive hypothesis, let $n \geq 6$ and suppose that for every tree T of order at least 3 and less than n the result is true. Let T be a tree of order $n \geq 6$. If $\text{diam}(T) = 2$, then T is a star, which yields $\gamma_{StR}(T) = \lceil \frac{n+1}{2} \rceil < (6n)/7$ by Proposition 15. If $\text{diam}(T) = 3$, then T is a double star and the result follows from Lemma 16. If T is a path, then by Observation 1 and Theorem A we have $\gamma_{StR}(T) = \gamma_R(T) \leq \frac{4n}{5} < \frac{6n}{7}$. Thus, we may assume that $\text{diam}(T) \geq 4$ and $\Delta(T) \geq 3$. For a subtree T' with n' vertices, where $n' \geq 3$, the induction hypothesis yields a StRDF f' of T' with weight at most $\frac{6n'}{7}$. We shall find a subtree T' such that adding a bit more weight to f' will yield a small enough StRDF f for T. Let $P = v_1 v_2 \ldots v_k$ be a diametral path in T chosen to maximize

$t = \deg_T(v_2)$. Also suppose among paths with this property we choose a path such that $|L_{v_3}|$ is as large as possible. Root T at v_k. We consider three cases.

Case 1: $t \geq 4$.
Let $T' = T - T_{v_2}$. Since $\operatorname{diam}(T) \geq 4$, we have $n' \geq 3$. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v_2) = 1 + \lceil \frac{t}{2} \rceil$ and $f(x) = 0$ for each $x \in L_{v_2}$. Note that f is a StRDF for T and that

$$w(f) = w(f') + 1 + \left\lceil \frac{t}{2} \right\rceil \leq \frac{6(n-t)}{7} + 1 + \left\lceil \frac{t}{2} \right\rceil < \frac{6n}{7}.$$

By the choice of the diametral path, it remains only to consider those trees, where all end-stems on diametral paths have degree at most 3.

Case 2: $t = 3$.
Let $L_{v_2} = \{v_1, u\}$. We consider the following subcases regarding the degree of the non leaf neighbor of v_2 in the diametral path P.

Subcase 2.1. $\deg_T(v_3) = 2$.
Let $T' = T - T_{v_3}$. If $|V(T')| = 2$, then $|V(T)| = 6$ and it is easy to see that $\gamma_{StR}(T) = 5 < \frac{6n}{7}$. Suppose that $|V(T')| \geq 3$. Define f on $V(T)$ by $f(x) = f'(x)$ for every $x \in V(T')$, $f(v_3) = 1, f(v_2) = 2$ and $f(x) = 0$ for $x \in L_{v_2}$. Clearly f is a StRDF for T of weight $w(f') + 3$ and so,

$$w(f) = w(f') + 3 \leq \frac{6(n-4)}{7} + 3 < \frac{6n}{7}.$$

Subcase 2.2. $\deg_T(v_3) \geq 3$ and v_3 is adjacent to an end-stem w of degree 3 such that $w \neq v_2$. Assume that $L_w = \{w_1, w_2\}$ and let $T' = T - \{v_2, v_1, u, w, w_1, w_2\}$. If $|V(T')| = 2$, then $|V(T)| = 8$ and it is easy to see that $\gamma_{StR}(T) = 6 < \frac{6n}{7}$. Suppose $|V(T')| \geq 3$. Define f on $V(T)$ by $f(x) = f'(x)$ for every $x \in V(T') - \{v_3\}$, $f(v_3) = f'(v_3) + 1$, $f(v_2) = f(w) = 2$ and $f(x) = 0$ for $x \in L_w \cup L_{v_3}$. Obviously f is a StRDF for T of weight $w(f') + 5$, and so

$$w(f) = w(f') + 5 \leq \frac{6(n-6)}{7} + 5 < \frac{6n}{7}.$$

By Subcase 2.2, we only need to consider the possibilities in which all end-stems adjacent to v_3, with exception v_2, have degree 2. If $\operatorname{diam}(T) = 4$, then $T - T_{v_2}$ is a spider, and it is easy to see that $\gamma_{StR}(T) \leq \frac{6n}{7}$. Let $\operatorname{diam}(T) \geq 5$

Subcase 2.3. $\deg_T(v_3) \geq 4$ is an even number.
Let $T' = T - D(v_3)$. Hence, v_3 is a leaf in T' and also $|f'(v_3)| \geq 1$ or $|f'(v_4)| \geq 1$. Define f on $V(T)$ by assigning $f(x) = f'(x)$ for every $x \in V(T') - \{v_3\}$, $f'(v_3) + \left\lfloor \frac{\deg(v_3)}{2} \right\rfloor$ to v_3, 2 to v_2, 1 to the leaf adjacent to an end-stem of degree 2 in T_{v_3}, and 0 to the remaining vertices. Notice that f is a StRDF for T of weight $w(f') + \left\lfloor \frac{\deg(v_3)}{2} \right\rfloor + 2 + r$, where r is the number of end-stems of degree 2 in T_{v_3}. Since $|D(v_3)| = \deg(v_3) + r + 1$, we deduce that

$$w(f) \leq \frac{6(n - |D(v_3)|)}{7} + \left\lfloor \frac{\deg(v_3)}{2} \right\rfloor + 2 + r < \frac{6n}{7}.$$

Subcase 2.4. $\deg_T(v_3) \geq 5$ is odd.
Let $T' = T - T_{v_3}$. Define f on $V(T)$ by assigning $f(x) = f'(x)$ for every $x \in V(T')$, $\left\lceil \frac{\deg(v_3)}{2} \right\rceil$ to v_3,

2 to v_2, 1 to the leaf adjacent to an end-stem of degree 2 in T_{v_3}, and 0 to the remaining vertices. Observe that f is a StRDF for T of weight $w(f') + \left\lceil \frac{\deg(v_3)}{2} \right\rceil + 2 + r$, where r is the number of end-stems of degree 2 in T_{v_3}. Since $n = |V(T')| + \deg(v_3) + r + 2$, we deduce that

$$w(f) \leq \frac{6(n - \deg(v_3) - r - 2)}{7} + \left\lceil \frac{\deg(v_3)}{2} \right\rceil + 2 + r < \frac{6n}{7}.$$

Subcase 2.5. $\deg(v_3) = 3$ and v_3 is adjacent to an end-stem w of degree 2.
Suppose w' is the leaf adjacent to w. Let $T' = T - T_{v_3}$. Define f on $V(T)$ by $f(x) = f'(x)$ for every $x \in V(T')$, $f(v_3) = f(v_2) = 2$, $f(w') = 1$, and $f(v_1) = f(u) = 0$. Clearly, f is a StRDF for T of weight $w(f') + 5$ and so,

$$w(f) = w(f') + 5 \leq \frac{6(n-6)}{7} + 5 < \frac{6n}{7}.$$

Subcase 2.6. $\deg(v_3) = 3$ and v_3 is adjacent to a leaf w.
Suppose $T' = T - T_{v_3}$ and define f on $V(T)$ by $f(x) = f'(x)$ for every $x \in V(T')$, $f(v_3) = f(v_2) = 2$, and $f(v_1) = f(u) = f(w) = 0$. Obviously, f is a StRDF for T of weight $w(f') + 4$ and as above

$$w(f) = w(f') + 4 \leq \frac{6(n-5)}{7} + 4 < \frac{6n}{7}.$$

Case 3: $t = 2$.
By the choice of the diametral path, we only need to consider those possibilities in which every end-stem on a diametral path has degree 2. In particular, when every end-stem adjacent to v_3 has degree 2. Thus, it follows that T_{v_3} is a spider. Assume that $\delta_T(v_3) = d$ and r is the number of end-stems adjacent to v_3 in T_{v_3}.

First let $\mathrm{diam}(T) = 4$. Hence, T is a spider obtained from a star $K_{1,d}$ by subdividing $r+1$ edges where $2 \leq r+1 \leq d$. So $|V(T)| = n = d + r + 2$. Define $f : V(T) \to \{0, \ldots, 1 + \lceil \frac{d}{2} \rceil\}$ by $f(v_3) = 1 + \lceil \frac{d}{2} \rceil$, $f(x) = 0$ for every $x \in N(v_3)$ and $f(x) = 1$ otherwise. Clearly, f is a StRDF on G of weight $r + \lceil \frac{d}{2} \rceil + 2$. If d is even, then it is easy to see that $\gamma_{StR}(T) \leq r + \lceil \frac{d}{2} \rceil + 2 < \frac{6n}{7}$. If d is odd and $r + 1 \leq d - 1$, then we obtain $\gamma_{StR}(T) \leq r + \frac{d+1}{2} + 2 < \frac{6n}{7}$. Assume that d is odd and $d = r + 1$. Thus, $\gamma_{StR}(T) \leq r + \frac{d+1}{2} + 2 \leq \frac{6n}{7}$ with equality if and only if $d = r + 1 = 3$ and this happen if and only if $T = S(K_{1,3})$.

Now let $\mathrm{diam}(T) \geq 5$. Consider the following subcases.

Subcase 3.1. $\deg_T(v_3) \geq 3$. We distinguish some possibilities.

(a) $\deg_T(v_3) \geq 4$ is even.

Let $T' = T - T_{v_3}$. Define $f : V(T) \to \{0, \ldots, 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil\}$ by assigning $f(x) = f'(x)$ for every $x \in V(T')$, $1 + \frac{d}{2}$ to v_3, 1 to leaves at distance 2 from v_3 and 0 to the vertices in $N(v_3) - \{v_4\}$. We notice that f is an StRDF of T of weight $\omega(f') + 1 + \frac{d}{2} + r$. By the induction hypothesis we obtain

$$w(f) \leq \frac{6(n - d - r)}{7} + 1 + \frac{d}{2} + r \leq \frac{6n}{7},$$

with equality if and only if $d = 4$ and $r = 3$.

(b) $\deg_T(v_3) \geq 7$ is odd.

Let $T' = T - T_{v_3}$. Define $f : V(T) \to \{0, \dots, 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil\}$ by assigning $f(x) = f'(x)$ for every $x \in V(T')$, $1 + \frac{d+1}{2}$ to v_3, 1 to leaves at distance 2 from v_3 and 0 to the vertices in $N(v_3) - \{v_4\}$. Obviously f is an StRDF of T of weight $\omega(f') + 1 + \frac{d+1}{2} + r$, and it follows from the induction hypothesis that

$$w(f) \leq \frac{6(n-d-r)}{7} + 1 + \frac{d+1}{2} + r < \frac{6n}{7}.$$

(c) $\deg_T(v_3) = 5$.

- $r = 4$. Assume that $N(v_3) \setminus \{v_2, v_4\} = \{w_1, w_2, w_3\}$ and let w_i' be the leaf adjacent to w_i for each $i = 1, 2, 3$. Let $T' = T - \{v_1, v_2, w_1, w_1', w_2', w_3'\}$. Hence, either $f'(v_4) = 0$ and $|f'(v_3)| + |f'(w_2)| + |f'(w_3)| \geq 3$, or $f'(v_4) \neq 0$ and $|f'(v_3)| + |f'(w_2)| + |f'(w_3)| \geq 2$. Define $f : V(T) \to \{0, \dots, 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil\}$ by $f(x) = f'(x)$ for every $x \in V(T') - \{v_3, w_2, w_3\}$, $f(v_3) = |f'(v_3)| + |f'(w_2)| + |f'(w_3)| + 1$, $f(x) = 0$ for every $x \in N(v_3) - \{v_4\}$ and $f(x) = 1$ otherwise. It is easy to see that f is an StRDF of T of weight $\omega(f') + 5$ and, by the induction hypothesis, we have $w(f) \leq \frac{6(n-6)}{7} + 5 < \frac{6n}{7}$.
- $r = 3$. Let $L_{v_3} = \{w_3\}$, $N(v_3) \setminus (\{v_2, v_4\} \cup L_{v_3}) = \{w_1, w_2\}$ and w_i' be the leaf adjacent to w_i for each $i = 1, 2$. Suppose that $T' = T - \{v_1, v_2, w_1, w_1', w_2'\}$. Hence, either $f'(v_4) = 0$ and $|f'(v_3)| + |f'(w_3)| + |f'(w_3)| \geq 3$, or $f'(v_4) \neq 0$ and $|f'(v_3)| + |f'(w_2)| + |f'(w_3)| \geq 2$. The function f defined as [(a)], is clearly an StRDF of T of weight $\omega(f') + 4$ and, as above we have $w(f) \leq \frac{6(n-5)}{7} + 4 < \frac{6n}{7}$.
- $r = 2$. Suppose that $L_{v_3} = \{w_2, w_3\}$ and $N(v_3) \setminus (\{v_2, v_4\} \cup L_{v_3}) = \{w_1\}$ and w_1' is the leaf adjacent to w_1. Let $T' = T - \{v_1, v_2, w_1, w_1'\}$. Using an argument similar to that described in case [(a)], we obtain $w(f) < \frac{6n}{7}$.
- $r = 1$. Let $L_{v_3} = \{w_1, w_2, w_3\}$ and $T' = T - \{v_1, v_2, w_1\}$. An argument similar to that described in [(a)] shows that $w(f) < \frac{6n}{7}$.

(d) $\deg_T(v_3) = 3$.

Let $T' = T - T_{v_3}$. If $f'(v_4) \neq 0$, then define $f : V(T) \to \{0, \dots, 1 + \left\lceil \frac{\Delta(T)}{2} \right\rceil\}$ by assigning $f(x) = f'(x)$ to every $x \in V(T')$, 2 to v_3, 1 to the leaves at distance 2 from v_3 in T_{v_3}, and 0 to the remaining vertices. If $f'(v_4) = 0$, then define f by assigning $f(x) = f'(x)$ to every $x \in V(T')$, 2 to the end-stems adjacent to v_3, 1 to leaf adjacent to v_3 if any, and 0 to the remaining vertices. Observe that f is a StRDF of T of weight $\omega(f') + |V(T_{v_3})| - 1$. Since $|V(T_{v_3})| = 4$ or 5, we deduce from the induction hypothesis that

$$\gamma_{StR}(T) \leq \frac{6n - 6|V(T_{v_3})|}{7} + |V(T_{v_3})| - 1 < \frac{6n}{7}.$$

Subcase 3.2. $\deg_T(v_3) = 2$.

By the choice of the diametral path, we only need to consider the case in which all vertices adjacent

to v_4 with depth 2, have degree 2 and also, by symmetry, we may assume $\deg(v_{k-1}) = \deg(v_{k-2}) = 2$. Using an argument similar to that described in Case 1, we may assume all end-stems adjacent to v_4 have degree at most 3.

(a) $\deg(v_4) = 2$.
Let $T' = T - T_{v_4}$. If $|V(T')| = 2$, then $T = P_6$ and the result is immediate. Let $|V(T')| \geq 3$. If $f'(v_5) = 0$, then define f on T by $f(x) = f'(x)$ for every $x \in V(T')$, $f(v_4) = f(v_2) = 0$, $f(v_3) = 2$ and $f(v_1) = 1$. Also, if $f'(v_5) \neq 0$, then define f on T by $f(x) = f'(x)$ for every $x \in V(T')$, $f(v_3) = f(v_1) = 0$, $f(v_2) = 2$ and $f(v_4) = 1$. Note that f is an StRDF of T of weight $\omega(f') + 3$ and by the induction hypothesis we have $\gamma_{StR}(T) \leq \frac{6(n-4)}{7} + 3 < \frac{6n}{7}$.

(b) $\deg(v_4) \geq 3$ and v_4 is adjacent to an end-stem of degree 3, say w.
Let w_1, w_2 be the leaves adjacent to w and $T' = T - \{v_1, v_2, v_3, w, w_1, w_2\}$. If $f'(v_4) = 0$, then define f on T as $f(x) = f'(x)$ for every $x \in V(T')$, $f(w_2) = 1, f(w_1) = f(v_2) = 2$, $f(w) = f(v_3) = f(v_1) = 0$. Also, if $f'(v_4) \geq 1$, then define f on T as $f(x) = f'(x)$ for every $x \in V(T')$, $f(v_1) = 1, f(v_3) = f(w) = 2$, $f(w_1) = f(w_2) = f(v_2) = 0$. Again, we observe that f is an StRDF of T of weight at most $\omega(f') + 5$ and, by the induction hypothesis, we have $\gamma_{StR}(T) \leq \frac{6(n-6)}{7} + 5 < \frac{6n}{7}$.

(c) $\deg(v_4) \geq 3$ and v_4 is adjacent to an end-stem of degree 2, say w.
Let w' be the leaf adjacent to w. Let $T' = T - \{v_1, v_2, v_3, w, w'\}$. If $f'(v_4) \leq 1$, then define f on T as $f(x) = f'(x)$ for every $x \in V(T')$, $f(w') = f(v_2) = 2$, $f(w) = f(v_3) = f(v_1) = 0$. Also, if $f'(v_4) \geq 2$, then define f on T by $f(x) = f'(x)$ for every $x \in V(T') - \{v_4\}$, $f(v_4) = f'(v_4) + 1$, $f(w') = 1, f(v_2) = 2$, $f(w) = f(v_3) = f(v_1) = 0$. Clearly, f is an StRDF of T of weight at most $\omega(f') + 5$ and, by the induction hypothesis, we have $\gamma_{StR}(T) \leq \frac{6(n-6)}{7} + 5 < \frac{6n}{7}$.

(d) $\deg(v_4) \geq 3$ and there is a path $v_4w_3w_2w_1$ in T such that $w_3 \notin \{v_3, v_5\}$.
Hence, we must have $\delta(w_3) = \delta(w_2) = 2$ and $\delta(w_1) = 1$. Let $T' = T - \{v_i, w_i \mid 1 \leq i \leq 2\}$. If $f'(v_4) \leq 1$, then define f on T as $f(x) = f'(x)$ for every $x \in V(T')$, $f(w_2) = f(v_2) = 2$, $f(w_3) = f(v_3) = f(v_1) = f(w_1) = 0$. Also, if $f'(v_4) \geq 2$, then define f on T as $f(x) = f'(x)$ for $x \in V(T') - \{v_4\}$, $f(v_4) = f'(v_4) + 1$, $f(w_2) = f(v_2) = 2$, $f(w_3) = f(v_3) = f(v_1) = f(w_1) = 0$. Notice that f is an StRDF of T of weight at most $\omega(f') + 4$ and, by the induction hypothesis, we have $\gamma_{StR}(T) \leq \frac{6(n-5)}{7} + 4 < \frac{6n}{7}$.

(e) $\deg(v_4) = 3$ and v_4 is adjacent to a leaf, say w.
Hence, clearly $\mathrm{diam}(T) \geq 6$. Let $T' = T - T_{v_4}$. If $f'(v_4) \geq 1$, then define f on T by $f(x) = f'(x)$ for every $x \in V(T')$, $f(v_4) = f(v_2) = 2$, $f(w) = f(v_3) = f(v_1) = 0$. Also, if $f'(v_4) = 0$, then define f on T by $f(x) = f'(x)$ for every $x \in V(T')$, $f(w) = f(v_2) = 2$, $f(v_4) = f(v_3) = f(v_1) = 0$. Obviously f is an StRDF of T of weight at most $\omega(f') + 4$ and, by the induction hypothesis, we have $\gamma_{StR}(T) \leq \frac{6(n-5)}{7} + 4 < \frac{6n}{7}$.

(f) $\deg(v_4) \geq 4$ and every neighbor of v_4 but v_3, v_5 is a leaf.
Clearly $\mathrm{diam}(T) \geq 6$. Let $T' = T - T_{v_4}$ and define f on T by $f(x) = f'(x)$ for every $x \in V(T')$, $f(v_4) = 1 + \left\lceil \frac{\deg(v_4)}{2} \right\rceil$, $f(v_2) = 2$ and $f(x) = 0$ otherwise. Note that f is an StRDF of T of weight $\omega(f') + 3 + \left\lceil \frac{\deg(v_4)}{2} \right\rceil$. Using the induction hypothesis we can check that $\gamma_{StR}(T) < \frac{6n}{7}$.

This completes the proof. $\square$

Let $S(K_{1,3})$ (the star $K_{1,3}$ with all its edges subdivided) be rooted in its center v and let F_m^p consist of all the rooted product graphs $T \circ_v S(K_{1,3})$, where T is any tree on m vertices (see Figure 7 for an example). We notice that, if the graph $H = S(K_{1,3})$ is an induced subgraph in a graph G, and its noncentral vertices have no neighbors outside H in G, then any StRDF must put total weight at least 6 on the vertices of H. For the case of trees $T \in F_m^p$, they contain m disjoint induced subgraphs isomorphic to $S(K_{1,3})$ satisfying the situation mentioned above. So, $\gamma_{StR}(T) \geq 6|V(T)|/7$ for each $T \in F_m^p$. Clearly, the trees belonging to F_m^p are dependent of the rooted tree T' used in the rooted product $T' \circ_v S(K_{1,3})$, and this help us to characterize the trees achieving equality in Theorem 17.

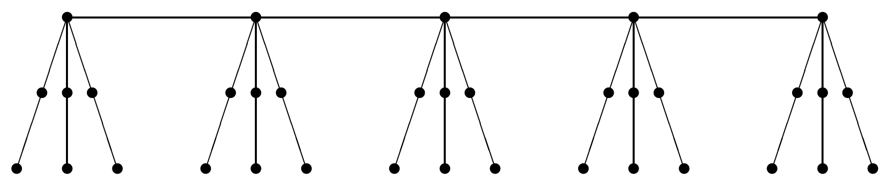


Figure 7: A member of F_5^p

Theorem 18. *Let T be an n-vertex tree. Then $\gamma_{StR}(T) = 6n/7$ if and only if $T \in F_m^p$.*

Proof. We have seen that if an induced subgraph H of G is isomorphic to $S(K_{1,3})$, and its noncentral vertices have no neighbors outside H in G, then every StRDF of G must put weight at least 6 on $V(H)$. Since every tree $T \in F_m^p$ has a vertex partition of m sets inducing such a subgraphs, the weight at least 6 is needed on every set of such partition. Moreover, it is easy to find a $\gamma_{StR}(T)$-function of weight $6n/7$, which leads to the equality.

To prove that equality requires this structure, we examine the proof of Theorem 17 more closely. We proceed by induction on n. In the base cases, $\mathrm{diam}(T) \leq 3$ and Cases 1 and 2, we produce an StRDF of weight less than $6n/7$. In Case 3 with diameter 4, equality requires $T = S(K_{1,3})$. Let $\mathrm{diam}(T) \geq 5$ and let $v_1 v_2 \dots v_k$ be a diametral path in T. Root T at v_k and let $T' = T - T_{v_3}$. Since $\gamma_{StR}(T) = 6n/7$, we deduce that $T_{v_3} = S(K_{1,3})$ and $\gamma_{StR}(T') = \frac{6|V(T_1)|}{7}$. By the induction hypothesis, $V(T_1)$ can be partitioned into sets inducing $S(K_{1,3})$ such that the subgraph induced by the central vertices of these subdivision stars is connected. Suppose $\{v, u_1, u_2, u_3, w_1, w_2, w_3\}$ is the partition set inducing $S(K_{1,3})$ with central vertex v and leaves u_1, u_2, u_3 containing v_4 in which w_i is the support vertex of u_i for each i. We claim that $v_4 = v$. Otherwise, we may assume without loss of generality that $v_4 \in \{u_1, u_2, u_3, w_1, w_2, w_3\}$. If $v_4 \in \{u_1, u_2, u_3\}$, then define $f : V(T) \to \{0, 1, 2, 3\}$ by $f(v_4) = f(w_1) = f(w_2) = f(w_3) = 0, f(v) = 3, f(x) = 1$ for every $x \in \{u_1, u_2, u_3\} - \{v_4\}$ and let f assign 3 to all other central vertices, 0 to all neighbors of central vertices and 1 to all leaves in each partition sets inducing $S(K_{1,3})$. If $v_4 \in \{w_1, w_2, w_3\}$, then define $f : V(T) \to \{0, 1, 2, 3\}$ by $f(w_1) = f(w_2) = f(w_3) = 0, f(v) = 2, f(x) = 1$ for $x \in \{u_1, u_2, u_3\}$ and let f assign 3 to all other central vertices, 0 to all neighbors of central vertices and 1 to all leaves in each partition sets inducing $S(K_{1,3})$. It is easy to see that in each case, f is a StRDF of T with weight less than $6n/7$ which is a contradiction. Thus v_4 is the central vertex of the subdivision vertex $S(K_{1,3})$ and the proof is completed. $\square$

In the sequel, we characterize all trees T of order $n \geq 3$ with $\gamma_{StR}(G) = n - 1$.

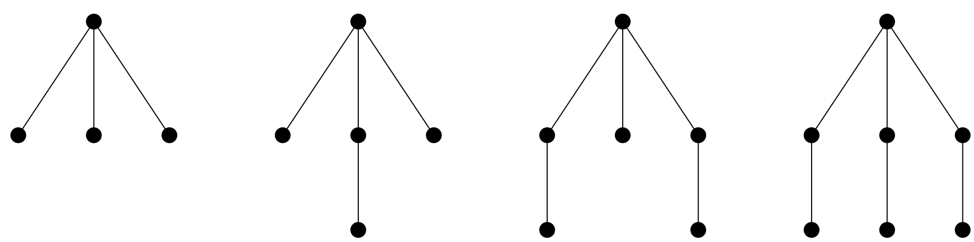
Lemma 19. *Let G be a connected graph of order n satisfying one of the following statement:*

1. *G has two vertices x and y of degree at least 2 with $d_G(x, y) \geq 3$.*
2. *G has two adjacent vertices x and y of degree at least three with $N(x) \cap N(y) = \emptyset$*
3. *G has two non-adjacent vertices x and y of degree at least three with $|N(x) \cap N(y)| \leq 1$.*

Then $\gamma_{StR}(G) \leq n - 2$.

Proof. Assume that $\{x_1, x_2\} \subseteq N(x) \setminus N[y]$ and $\{y_1, y_2\} \subseteq N(y) \setminus N[x]$. By assumption $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$. Then the function f given by $f(u) = 0$ for $u \in \{x_1, x_2, y_1, y_2\}$, $f(x) = f(y) = 2$ and $f(u) = 1$ otherwise, is a strong Roman dominating function of weight $w(f) = n - 6 + 4 = n - 2$ and hence $\gamma_{StR}(G) \leq n - 2$. $\square$

Let $\mathcal{T}$ be the family of trees consisting of the paths P_3, P_4, P_5 and the following four trees.



(a) (b) (c) (d)

Figure 8: The trees of the family $\mathcal{T}$.

Theorem 20. *Let T be a tree of order $n \geq 3$. Then $\gamma_{StR}(T) = n - 1$ if and only if $T \in \mathcal{T}$.*

Proof. If $T \in \mathcal{T}$, then it is easy to see that $\gamma_{StR}(T) = n - 1$.

Conversely, let $\gamma_{StR}(T) = n - 1$. By Proposition 9 and Theorem 17, we deduce that $\Delta(T) \leq 3$ and $n \leq 7$. If $\Delta(T) = 2$, then it follows from Proposition B and Observation 1 that $T \cong P_q$ for $q = 3, 4, 5$. Now, let $\Delta(T) = 3$. We deduce from Proposition 11 and Lemma 19 that $\mathrm{diam}(T) \leq 4$ and T has a unique vertex of degree 3. This implies that T is one one the trees in Figure and so $T \in \mathcal{T}$. $\square$

We conclude this section with an open problem.

Conjecture. For any connected graph G of order $n \geq 3$,

$$\gamma_{StR}(G) \leq 6n/7.$$

Moreover, we notice that if such abound will be true, then the equality holds if and only if G is obtained as the rooted product graph $G \circ_v S(K_{1,3})$, where G is any connected graph and $S(K_{1,3})$ is rooted in its center v.

5 Realizability for trees

It is clear that the maximum number of legions which are necessary to defend a graph G, under the strong Roman domination strategy, is the order of the graph. In this sense, as usual when a new parameter is introduced, it would be interesting to know if there exist graphs of order n, achieving all the possible suitable values for the strong Roman domination number. That is, in concordance with Proposition 14, all the integer numbers in the interval $\left\{\left\lceil\frac{n+1}{2}\right\rceil, \dots, n\right\}$. Equivalently, we should deal with the problem of realization for the strong Roman domination. That is, given two positive integers n, p such that $\lceil\frac{n+1}{2}\rceil \le p \le n$: Is there a graph of order n and strong Roman domination number equal to p? Next we partially solve this problem in concordance with the conjecture presented at the end of the section above. To this end, we need the following lemma, where we give the strong Roman domination number of spiders.

Lemma 21. *If T is a spider obtained from $K_{1,t}$ $(t \ge 2)$ by subdividing q edges $(0 \le q \le t)$, then*

$$\gamma_{StR}(T) = 1 + q + \left\lceil\frac{t}{2}\right\rceil.$$

Proof. If $t = 2$, then T is a path and the result is immediate by Theorem B and Observation 1. If $t = 3$, then the result follows from Theorem 20. Let $t \ge 4$. By Corollary 15, we may assume that $q \ge 1$. Let v be the central vertex of $K_{1,t}$ and let $N(v) = \{v_1, \dots, v_t\}$. Suppose u_i is the subdivision vertex of the edge vv_i for $1 \le i \le q$. Defined $f : V(T) \to \{0, 1, \dots, 1 + \lceil\frac{t}{2}\rceil\}$ by $f(v) = 1 + \lceil\frac{t}{2}\rceil$, $f(v_i) = 1$ for $1 \le i \le q$ and $f(x) = 0$ otherwise. It is easy to see that f is a strong Roman dominating function of T of weight $1 + q + \lceil\frac{t}{2}\rceil$ and hence $\gamma_{StR}(T) \le 1 + q + \lceil\frac{t}{2}\rceil$.

Now we show that $\gamma_{StR}(T) \ge 1 + q + \lceil\frac{t}{2}\rceil$. Let f be a $\gamma_{StR}(T)$-function such that $f(v)$ is as large as possible. If $f(v) \le 1$, then clearly $\omega(f) \ge n-1$ that leads to a contradiction with Theorem 20. Let $f(v) \ge 2$. If $f(u_i) \ge 1$ for some $1 \le i \le q$, then the function $g : V(T) \to \{0, 1, \dots, 1 + \lceil\frac{t}{2}\rceil\}$ defined by $g(v) = f(v) + 1, g(u_i) = 0, g(v_i) = 1$ and $g(x) = f(x)$ otherwise, is a $\gamma_{StR}(T)$-function that contradicts the choice of f. If $f(v_j) \ge 1$ for some $q + 1 \le j \le t$, then the function $g : V(T) \to \{0, 1, \dots, 1 + \lceil\frac{t}{2}\rceil\}$ defined by $g(v) = f(v) + 1, g(v_j) = 0$ and $g(x) = f(x)$ otherwise, is a $\gamma_{StR}(T)$-function that contradicts the choice of f again. Thus $f(v_i) = 0$ for each $1 \le i \le t$ implying that $f(v) \ge 1 + \lceil\frac{t}{2}\rceil$ and $f(v_i) = 1$ for $1 \le i \le q$. Hence $\gamma_{StR}(T) \ge 1 + q + \lceil\frac{t}{2}\rceil$ and the proof is complete. ∎

We also need the next result, where we compute the strong Roman domination number of some family of graphs $\widetilde{\mathcal{F}}$ defined as follows. A graph $G_n(q, j, l) \in \widetilde{\mathcal{F}}$ if and only if $G_n(q, j, l)$ is obtained as the rooted product graph $G(\mathcal{H})$, where $G = v_1v_2...v_{q+1}$ is a path on $q + 1$ vertices and the ordered family $\mathcal{H} = \{S_{j+l,j}, S_{3,3}, S_{3,3}, ..., S_{3,3}\}$ is formed by one (wounded or healthy) spider $S_{j+l,j}$ and q healthy spiders $S_{3,3}$ having their roots in their centers. See Figure 9 for an example.

Lemma 22. *For any graph $G_n(q, j, l) \in \widetilde{\mathcal{F}}$, $\gamma_{StR}(G_n(q, j, l)) = 6q + \lceil\frac{j+l}{2}\rceil + j + 1$.*

Proof. By Lemma 21 we have that $\gamma_{StR}(S_{j+l,j}) = 1 + j + \lceil\frac{j+l}{2}\rceil$. Also, by Lemma 18 we know that the subgraph G' of $G_n(q, j, l)$ induced by the q healthy spiders $S_{3,3}$ satisfies that $\gamma_{StR}(G') = 6q$. Clearly, a function f in $G_n(q, j, l)$ obtained from a combination of a $\gamma_{StR}(S_{j+l,j})$-function f_1 and a $\gamma_{StR}(G')$-function f_2, is a strong Roman dominating function in $G_n(q, j, l)$. Thus, we have that $\gamma_{StR}(G_n(q, j, l)) \le w(f_1) + w(f_2) = 6q + \lceil\frac{j+l}{2}\rceil + j + 1$.

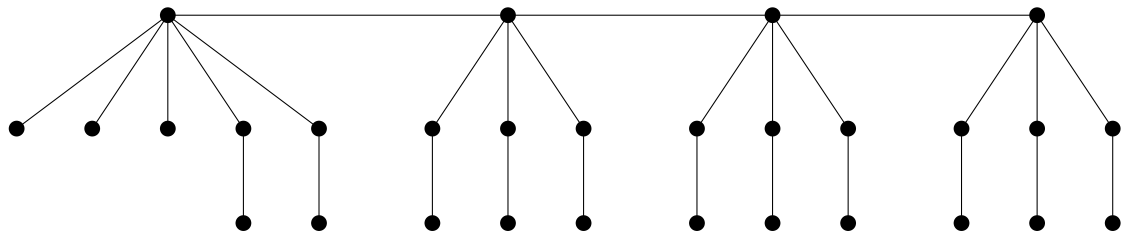


Figure 9: The graph $G_{29}(3,2,3)$ of the family $\widetilde{\mathcal{F}}$.

On the other hand, an argument similar to that described in the proof of Lemma 21 can be used to obtain the lower bound. Since the induced subgraph $S_{j+l,j}$ of $G_n(q,j,l)$ is isomorphic to a spider, and its noncentral vertices have no neighbors outside of $S_{j+l,j}$, every StRDF of $G_n(q,j,l)$ must put weight at least $1+q+\lceil\frac{j+l}{2}\rceil$ on $V(S_{j+l,j})$. Similarly, every StRDF of $G_n(q,j,l)$ must put weight at least $6q$ in the subgraph G' of $G_n(q,j,l)$ induced by the q healthy spiders $S_{3,3}$. Therefore, $\gamma_{StR}(G_n(q,j,l)) \geq 6q+\lceil\frac{j+l}{2}\rceil+j+1$ and the proof is completed. □

Once given the value for the strong Roman domination of graphs of the family $\widetilde{\mathcal{F}}$ we are able to present our realizability result for trees.

Theorem 23. *Let n,p be any two integers such that $n \geq 3$ and $\lceil\frac{n+1}{2}\rceil \leq p \leq \lfloor\frac{6n}{7}\rfloor$. Then, there exists a graph G of order n with $\gamma_{StR}(G)=p$.*

Proof. If $n=3$, then $\lceil\frac{n+1}{2}\rceil \leq \lfloor\frac{6n}{7}\rfloor = 2$. So $p=2$. Hence, a possible graph for this case would be the path P_3. If $n=4$, then $\lceil\frac{n+1}{2}\rceil = \lfloor\frac{6n}{7}\rfloor = 3$. Thus, it must happen $p=3$ and a possibility for the realization of a graph would be the star graph $S_{1,3}$.

From now on we consider $n \geq 5$ and for these cases, we shall use the graphs of the family $\widetilde{\mathcal{F}}$ for some specific values of q,j,l, satisfying that $0 \leq q \leq \lfloor\frac{n}{7}\rfloor$, $0 \leq j \leq 4$ and $j+l \geq 3$ (we recall that the condition $0 \leq j \leq 4$ is introduced for our purposes on the existence of a graph realizing the values of n and p). From Lemma 22, we know that a graph $G_n(q,j,l) \in \widetilde{\mathcal{F}}$ has order n and $\gamma_{StR}(G_n(q,j,l)) = 6q+\lceil\frac{j+l}{2}\rceil+j+1$. Thus, given the integers n,p as described in the statement, it only remains to find the suitable values of q,j,l in terms of n,p, or equivalently, to obtain the solutions of the following systems of equations.

$$\begin{cases} n = 7q+2j+l+1 \\ p = 6q+\lceil\frac{j+l}{2}\rceil+j+1 \end{cases} \tag{2}$$

We proceed with the solution of the system above, for q and l, according to the parity of the number $j+l$ and using the fact that $0 \leq j \leq 4$. If $j+l$ is even, then the system (2) becomes

$$\begin{cases} n = 7q+2j+l+1 \\ 2p = 12q+3j+l+2 \end{cases} \tag{3}$$

whose solution (for q and l) in terms of n,p,j is given by

$$q = \frac{2p-n-j-1}{5} \quad \text{and} \quad l = \frac{12n-14p-3j+2}{5}. \tag{4}$$

If $j+l$ is odd, then the system (2) becomes

$$\begin{cases} n = 7q + 2j + l + 1 \\ 2p = 12q + 3j + l + 3 \end{cases} \tag{5}$$

whose solution (for q and l) in terms of n, p, j is given by

$$q = \frac{2p - n - j - 2}{5} \text{ and } l = \frac{12n - 14p - 3j + 9}{5}. \tag{6}$$

We check now that these solutions are giving suitable values for q, l, according to the fact that $0 \le j \le 4$, in order to construct the graph $G_n(q, j, l)$. That is, we must check that integer solutions to the systems in (3) and (5) are possible. Consider the solution in (4). There we have that $2p - n = 5q + j + 1$. Since $p \ge \frac{n+1}{2}$, it follows that $2p - n \ge 1$. Thus, $2p - n$ can always be represented as $5q + j + 1$ for some integer q and some $j \in \{0, ..., 4\}$. Let q', j' obtained in this way. We must check that q', j' lead to an integer solution for l in (4). That is,

$$\begin{aligned} l &= \frac{12n - 14p - 3j' + 2}{5} \\ &= n - \frac{7(2p - n) + 3j' - 2}{5} \\ &= n - \frac{7(5q' + j' + 1) + 3j' - 2}{5} \\ &= n - 7q' + 2j' + 1. \end{aligned}$$

Thus, l is an integer number and the construction of $G_n(q, j, l)$ is possible. An analogous process leads to integer solutions of the system in (5).

Now, in order to complete the proof, we observe that the solutions for q in (4) and (6), lead to $p \ge \frac{n+1}{2}$ and $p \ge \frac{n+2}{2}$, respectively, since $j, q \ge 0$. So, it must happen $p \ge \left\lceil \frac{n+1}{2} \right\rceil$. It remains to check only the consequences of the solution for l. To this end, we consider the following cases.

Case 1: $j = 0$. If l is even, then $j + l$ is even. Since $j + l \ge 3$ (by assumption) and l is even, we obtain that $l \ge 4$. Hence, from (4) we have $12n - 4p + 2 \ge 20$, which leads to $p \le \frac{6n-9}{7} < \frac{6n}{7}$. If l is odd, then $j+l$ is odd and $l \ge 3$. From (4) we have $12n-4p+2 \ge 15$, which leads to $p \le \frac{6n-9}{7} < \frac{6n}{7}$.

Case 2: $j \ne 0$. If $j+l$ is even, then we must consider the solutions in (4). If j is even, then l is even and $l \ge 2$ (since $j + l \ge 3$). So, from the solution for l in (4) we have that $12n - 14p - 3j + 2 \ge 10$, which means that $p \le \frac{12n-3j-8}{14} \le \frac{12n-14}{14} = \frac{6n}{7} - 1 < \frac{6n}{7}$. If j is odd, then l is odd and $l \ge 3$ (again since $j + l \ge 3$). Thus, the solution for l in (4) leads to $12n - 14p - 3j + 2 \ge 15$, which gives $p \le \frac{12n-3j-13}{14} \le \frac{12n-16}{14} < \frac{6n}{7} - 1 < \frac{6n}{7}$. Finally, if $j + l$ is odd, then a similar process as above, by considering the parity of j, gives that always $p \le \frac{6n}{7}$, which completes the proof. $\square$

As a conclusion of the proof above, given two integers n, p such that $n \ge 3$ and $\left\lceil \frac{n+1}{2} \right\rceil \le p \le \left\lfloor \frac{6n}{7} \right\rfloor$, the construction of a graph G of order n with $\gamma_{StR}(G) = p$ is made in the following way.

- Find a value q' such that $2p - n = 5q' + j' + 1$ for some $j' \in \{0, ..., 4\}$.
- Compute the value $l' = n - 7q' + 2j' + 1$.
- Construct the desired graph G as the graph $G_n(q', j', l')$.

Conclusions

In this article we have introduced a new invariant related to domination in graphs, called the strong Roman domination number and denoted $\gamma_{StR}(G)$, for any graph G. We have proved that the decision problem regarding the existence of a strong Roman dominating function of minimum cardinality belongs to the NP-complete complexity class. In concordance with this fact, we have obtained several lower and upper bounds for $\gamma_{StR}(G)$ of any connected graph G. Such bounds regard some parameters in graphs like for instance, the order, the diameter, the girth, among others. We have also obtained an interesting upper bound on the strong Roman domination number of trees, namely $\gamma_{StR}(T) \leq \left\lfloor \frac{6n}{7} \right\rfloor$ for any tree T of order n, and have characterized the families of trees achieving such a bound. In concordance with such bound, and with a lower bound involving the order of any graph G, that is $\gamma_{StR}(G) \geq \left\lceil \frac{n+1}{2} \right\rceil$, we have presented a realizability result which involves all the possible values between such bounds. Specifically, we have proved that given two integers n, p such that $n \geq 3$ and $\left\lceil \frac{n+1}{2} \right\rceil \leq p \leq \left\lfloor \frac{6n}{7} \right\rfloor$, there exists a tree T of order n with $\gamma_{StR}(T) = p$.

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